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MTH 512, Exam 2

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QUESTION 1. Assume A, B are similar $n \times n$ matrices, say $A = M^{-1}BM$, and assume that A is diagnolizable. Prove that B is diagnolizable.

QUESTION 2. (Transitive property). Given A, B, C are $n \times n$ matrices such that A is similar to B and B is similar to C. Prove that A is similar to C, i.e., show that $A = N^{-1}CN$ for some invertible matrix N.

QUESTION 3. Let $A = \begin{bmatrix} 0 & -4 & 0 & 0 & 0 & 0 \\ 1 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 1 & -5 \end{bmatrix}$

(i) Find $A^{2023} + 5A^{2022} + 5A^{2021} + 5A^{2020} + 4A^{2019} + 3A^3 + 15A^2 + 13A + I_6$.[Hint: By staring at A, it looks familiar, so $C_A(\alpha)$ and $m_A(\alpha)$ can be determined]

(ii) Find all eigenvalues of A. For each eigenvalue a of A, find $dim(E_a(A))$.

QUESTION 4. Up to similarity, classify all 5×5 matrices in $R^{5\times 5}$ such that $(A^2 + I_5)(A - 3I_5) = 0_{5\times 5}$. [Hint : Note that all entries of such matrix must be real numbers]

QUESTION 5. Let $< , >$ be the normal dot product on R^k.

(i) Let $n \geq 2$ and $T : R^n \to R^n$ be a symmetric linear transformation, i.e., the standard matrix presentation of T is symmetric. Assume a, b are distinct eigenvalues of T. Let v be a nonzero-vector in $E_a(A)$ and w be a nonzero-vector in $E_b(A)$. Prove that v is orthogonal to w.

QUESTION 6. Let $T : R^2 \to P_2$ such that $T(a, b) = (b + 2a)x + 3a$. Define $< f_1, f_2 >_{P_2} = \int_0^1 f_1 f_2 \, dx$ and $< q_1, q_2 >_{R^2} = q_1 \cdot q_2$. Find T^a (the adjoint operator of T)

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Question 1:

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Assume A, B are similar $n \times n$ matrices, say $A = M^{-1}BM$ and assume that A is diagnolizable. Prove that B is diagnolizable.

A is diagnolizable, means $\exists$ non singular matrix Q, and diagonal matrix D s.t. $A = Q^{-1}DQ$

and $A = M^{-1}BM \iff MAM^{-1} = B$

thus $B = M(Q^{-1}DQ)M^{-1}$

$B = (MQ^{-1})D(QM^{-1})$

Now, since $(QM^{-1})^{-1} = (M^{-1})^{-1}Q^{-1} = MQ^{-1}$ (inverse is unique)

$B = (QM^{-1})^{-1} D (QM^{-1})$

QM^{-1} is an invertible matrix, D diagonal so B is diagnolizable.

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since A is diagnolizable, then the minimal polynomial is the product of linear factors.
and A is similar to B means, they have the same minimal polynomial as we proved in the Homework
thus the minimal polynomial of B is the product of linear factors. Hence B is diagnolizable.

Question 2: prove the transitive property.

suppose $A = Q^{-1} B Q$

and $B = P^{-1} C P$

P, Q are invertible matrices, then the Product PQ is also invertible

then $A = Q^{-1} (P^{-1} C P) Q$

$A = (Q^{-1} P^{-1}) C (PQ)$

and since $(PQ)^{-1} = Q^{-1} P^{-1}$

then $A = (PQ)^{-1} C (PQ)$

letting $PQ = N$ we get $A = N^{-1} C N$

that is A is similar to C.

Question 3:

i). $A = C(\alpha^2+5\alpha+4) \oplus C(\alpha^2+5\alpha+4) \oplus C(\alpha^2+5\alpha+4)$.

$C_A(\alpha) = (\alpha^2+5\alpha+4)(\alpha^2+5\alpha+4)(\alpha^2+5\alpha+4) = (\alpha+4)^3(\alpha+1)^3$

$m_A(\alpha)$ is the least common multiple of the three minimal polynomials

$m_A(\alpha) = (\alpha^2+5\alpha+4) = (\alpha+4)(\alpha+1)$

let $f(\alpha) = \alpha^{2023} + 5\alpha^{2022} + 5\alpha^{2021} + 5\alpha^{2020} + 4\alpha^{2019} + 3\alpha^3 + 15\alpha^2 + 13\alpha + 1$

then $f(\alpha) = \alpha^{2019}(\alpha^4 + 5\alpha^3 + 5\alpha^2 + 5\alpha + 4) + 3\alpha^3 + 15\alpha^2 + 13\alpha + 1$

$= \alpha^{2019}(\alpha^2+1)(\alpha^2+5\alpha+4) + 3\alpha(\alpha^2+5\alpha+4) + (\alpha+1)$

$f(\alpha) = \underbrace{(\alpha^2+5\alpha+4)}_{= 0_{6\times 6}}(\alpha^{2019}(\alpha^2+1) + 3\alpha) + \alpha + 1$

$f(\alpha) = \alpha + 1 \pmod{\alpha^2+5\alpha+4}$

Hence $f(A) = A + I_6 = \begin{bmatrix} 1 & -4 & 0 & 0 & 0 & 0 \\ 1 & -4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{bmatrix}$

ii). $m_A(\alpha) = (\alpha+4)(\alpha+1)$ (A is diagnolizable).

so the Eigen values of A are $\alpha_1 = -4$, $\alpha_2 = -1$

$\dim(E_{-4}(A)) = 3$

$\dim(E_{-1}(A)) = 3$

for each Eigen value the algebraic multiplicity equals the geometric multiplicity.

Question 4:

classify All real 5x5 matrices s.t. $\left(A^2 + I_5\right)(A - 3I) = 0_{5\times5}$

let $f(\alpha) = (\alpha^2 + 1)(\alpha - 3)$ [Note: since $\deg(C_A(\alpha)) = 5$ it must have a real root]

then $m_A(\alpha) \mid f(\alpha)$

so $m_A(\alpha) = (\alpha - 3)$ | or $m_A(\alpha) = (\alpha^2+1)(\alpha-3)$

so $C_A(\alpha) = (\alpha-3)^5$

$A = C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3) \oplus C(\alpha-3)$

= 3I_5

$C_A(\alpha) = (\alpha^2+1)^2(\alpha-3)^1$

let $f_1 = (\alpha^2+1)$

then

$A \approx C((\alpha^2+1)(\alpha-3)) \oplus C(\alpha^2+1)$

or $C_A(\alpha) = (\alpha^2+1)(\alpha-3)^3$

$f_2 = (\alpha-3)$

$f_1 = (\alpha-3)$

$A \approx C((\alpha^2+1)(\alpha-3)) \oplus C(\alpha-3) \oplus C(\alpha-3)$

note: $m_A(\alpha) \neq (\alpha^2+1)$ bcs A has at least one real eigen value and all eigen values of A are roots of $m_A(\alpha)$.

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Question 5:

$T: \mathbb{R}^n \to \mathbb{R}^n$ is a symmetric linear transformation.

let A be the standard matrix representation of T

A is symmetric $\Rightarrow$ $A = A^T$

a, b are distinct eigenvalues of T

$v \in E_a(A)$ so $Av = av$

$w \in E_b(A)$ so $Aw = bw$

We know that $T^a : \mathbb{R}^n \to \mathbb{R}^n$ s.t. $\forall v \in \mathbb{R}^n, w \in \mathbb{R}^n$

$$\langle T(v), w \rangle_{\mathbb{R}^n} = \langle v, T^a(w) \rangle_{\mathbb{R}^n}$$

since we are using the normal dot product on $\mathbb{R}^n$

if $T(v) = Av$ then $T^a(w) = A^T w$

Hence $\langle T(v), w \rangle = \langle v, T^a(w) \rangle$

$$\langle Av, w \rangle = \langle v, A^T w \rangle$$

$$\langle av, w \rangle = \langle v, Aw \rangle$$

$$a\langle v, w \rangle = \langle v, bw \rangle = \bar{b}\langle v, w \rangle$$

$$a\langle v, w \rangle = b\langle v, w \rangle$$

(since A is symmetric then all Eigen values of A are real)

$$(a-b)\langle v, w \rangle = 0$$

but $a \neq b$ thus $\langle v, w \rangle = 0$

since neither v, nor w are the zero vector we conclude they are orthogonal.

Question 6: $T: \mathbb{R}^2 \longrightarrow P_2$

$$T(a,b) = (b+2a)x + 3a$$

Define $\langle f_1, f_2 \rangle_{P_2} = \int_0^1 f_1 f_2 \, dx$ and $\langle q_1, q_2 \rangle_{\mathbb{R}^2} = q_1 q_2$

Find T^a (the adjoint operator of T).

Define,

$$T^a: P_2 \longrightarrow \mathbb{R}^2$$

$$T^a(cx+d) = (m,n)$$

$$\langle T(a,b) ; cx+d \rangle_{P_2} = \langle (a,b), (m,n) \rangle_{\mathbb{R}^2}$$

$$\langle (b+2a)x+3a, cx+d \rangle_{P_2} = am + bn$$

$$\frac{bc}{3} + \frac{bd}{2} + \frac{2ac}{3} + \frac{\cancel{2}ad}{\cancel{2}} + \frac{3ac}{2} + 3ad = am + bn$$

$$b\left[\frac{c}{3} + \frac{d}{2}\right] + a\left[\frac{2}{3}c + d + \frac{3}{2}c + 3d\right] = a(m) + b(n)$$

$$n = \frac{c}{3} + \frac{d}{2}$$

$$m = \frac{2}{3}c + d + \frac{3}{2}c + 3d = \frac{13}{6}c + 4d$$

Hence $T^a(cx+d) = \left(\frac{13}{6}c + 4d, \frac{c}{3} + \frac{d}{2}\right)$